

**On non σ -shortness of Axiom A
posets with frame systems**

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Previous Study

In this talk, we assume that Boolean algebras are atomless, posets are non-atomic and separative.

Definition

$D \subseteq B^+$ is *σ -short* if every strictly descending sequence of length ω in D does not have a nonzero lower bound in B .

B is said to be *σ -short* if it has a *σ -short* dense subset.

B is said to be *strongly σ -short* if it has a *σ -short* \wedge -closed dense subset.

D is \wedge -closed if and only if $x \wedge y \in D$ for every $x, y \in D$.

We note that B itself is not a σ -short set, since B is atomless.

σ -short posets are defined the same way as for BAs.

Examples of σ -short Boolean algebras

1. For any set X , let $\text{Fr } X$ be the free Boolean algebra over X .

$$D = \{\pm x_1 \cdot \pm x_2 \cdot \dots \cdot \pm x_n \mid n \in \omega, x_1, x_2, \dots, x_n \in X\} - \{0\}$$

Clearly, D is a σ -short \wedge -closed dense subset of $\text{Fr } X$. Hence, $\text{Fr } X$ is strongly σ -short.

Every regularly filtered Ba is also strongly σ -short[TY].

2. Let (\mathbf{B}, μ) be a measure algebra.

$$D = \{a \in \mathbf{B} \mid \mu(a) = \frac{1}{n+1} \text{ for some } n \in \omega\}.$$

D is a σ -short dense subset of \mathbf{B} . Hence (\mathbf{B}, μ) is σ -short.

Characterizations of σ -short BAs:

Q1. Is it true that the following are equivalent?

(1) B is σ -short.

(2) There exists a sequence $\{X_n\}_{n \in \omega}$ of subsets of B which satisfies the following conditions:

(a) X_n is a pairwise incomparable subset of B .

(b) If $x \in X_n, y \in X_m$ and $n < m$, then $y \not\leq x$.

(c) $X = \bigcup_{n \in \omega} X_n$ is a dense subset of B .

It holds that $(2) \Rightarrow (1)$.

Partial answer for strongly σ -short BAs

Theorem.([T]) The following are equivalent.

(1) B is strongly σ -short.

(2) There exists a sequence $\{X_n\}_{n \in \omega}$ of subsets of B which satisfies the following conditions:

(a) X_n is a pairwise incomparable subset of B .

(b) If $x \in X_n, y \in X_m$ and $n < m$, then $y \not\geq x$.

(c) $X = \bigcup_{n \in \omega} X_n$ is a dense subset of B

(d) $\{y \in X_n \mid y \geq x\}$ is finite for every $n \in \omega$ and $x \in X_{n+1}$.

Q.2 Is it true that every σ -short BAs are strongly σ -short?

No.

Theorem A(Brendle). Let B_κ be the algebra for adding κ many random reals.

(1) B_ω is not strongly σ -short.

(2) Assume that the density of B_κ equals to κ .

Then B_κ is strongly σ -short.

Partial answer to Q.1 for Axiom A posets:

Theorem 1. Let P be σ -short Axiom A poset. If P satisfies the conditions (C1), (C2) and (C3) (to be defined later), then there exists a sequence $\{X_n\}_{n \in \omega}$ of subsets of P which satisfies the the following conditions:

- (a) X_n is a pairwise incomparable subset of B .
- (b) If $x \in X_n, y \in X_m$ and $n < m$, then $y \not\leq x$.
- (c) $X = \bigcup_{n \in \omega} X_n$ is a dense subset of B .

Actually, many Axiom A posets are not σ -short.

A poset (P, \leq) satisfies Axiom A if there are partial orderings \leq_n ($n \in \omega$) such that

(A1): If $p \leq_0 q$ then $p \leq q$;

(A2): If $p \leq_{n+1} q$, then $p \leq_n q$;

(A3): If $\{p_n\}_{n \in \omega}$ is a fusion sequence; i.e., if $p_{n+1} \leq_n p_n$ for every $n \in \omega$, then there is q such that $q \leq_n p_n$ for all $n \in \omega$;

(A4): If $p \in P$ and W is a partition of p , then for every n there is $q \leq_n p$ such that q is compatible with at most countably many $x \in W$.

We say that a poset (P, \leq) with partial orderings \leq_n ($n \in \omega$) is a *fusion poset* if it satisfies (A1),(A2),(A3). We assume that partial orderings $\{\leq_n\}_{n \in \omega}$ are transitive.

First, we consider the following condition (C1) for fusion posets.

$$(C1): \forall n \in \omega \forall p \in P \exists p^* \geq_n p \forall p' \geq_n p [p^* \geq_n p']$$

For $n \in \omega$ and $p \in P$, we denote $*p$ in (C1) by *stem_n(p)*.

If a fusion poset \mathbf{P} satisfies (C1), then the relation \sim_n on \mathbf{P} defined by $p \sim_n q \stackrel{\text{def}}{\iff} stem_n(p) = stem_n(q)$ is an equivalence relation on \mathbf{P} .

Using this equivalence relation, we consider conditions (C2) and (C3) as follows.

$$(C2): \quad \forall n \in \omega [|\mathbf{P} / \sim_n| \leq \omega]$$

$$(C3): \quad \forall n \in \omega \forall p, q \in \mathbf{P} [p \sim_n q \ \& \ p \geq q \Rightarrow p \geq_n q]$$

Examples

In the following examples, we consider a canonical enumeration of $2^{<\omega}$ or $\omega^{<\omega}$. And, when we enumerate elements of a subset of those sets, we use this canonical enumeration.

If t appears in an enumeration after s , then we denote it by $s \prec t$.

Let $\omega^{<\omega \uparrow} = \{t \in \omega^{<\omega} \mid t \text{ is increasing}\}$.

Sacks forcing: (\mathbb{P}_S, \leq) is defined as follows.

$\mathbb{P}_S = \{p \mid p \text{ is a perfect tree of } 2^{<\omega}\}$ and $p \geq q$ iff $p \supseteq q$.

$p \geq_n q \Leftrightarrow p \geq q$ and $B_n(p) = B_n(q)$ where $B_n(p)$ is a set of the $(n + 1)$ -st branching points of p .

For $p \in \mathbb{P}_S$ and $n \in \omega$, put

$stem_n(p) = \{t \in 2^{<\omega} \mid \exists s \in B_n(p) [t \subseteq s \text{ or } s \subseteq t]\}$.

\mathbb{P}_S satisfies (C1). It holds that $p \sim_n q$ iff $B_n(p) = B_n(q)$.

So \mathbb{P}_S satisfies (C2) and (C3).

Laver forcing: (\mathbb{P}_L, \leq) is defined as follows.

$\mathbb{P}_L = \{p \mid p \text{ is a tree of } \omega^{<\omega} \text{ which has a stem } s \text{ such that}$
 $\forall t \supseteq s [S(t) = \{k \in \omega \mid t \frown k \in p\} \text{ is infinite}]\}$ and $p \geq q \Leftrightarrow p \supseteq q$.

For $p \in \mathbb{P}_L$, **let** $s_0^p = \text{stem}(p), s_1^p, \dots, s_m^p, \dots$ **be an enumeration of** $\{t \in p \mid t \supseteq \text{stem}(p)\}$.

$p \geq_n q$ **iff** $p \geq q$ **and** $s_i^p = s_i^q$ **for all** $i = 0, \dots, n + 1$.

For $p \in \mathbf{P}_L$ and $n \in \omega$,

$$\text{stem}_n(p) = \{t \in p \mid t \subseteq s_0^p\} \cup \{s_1^p, \dots, s_{n+1}^p\} \cup \{t \in \omega^{<\omega} \mid s_{n+1}^p \prec t\}.$$

\mathbf{P}_L satisfies (C1).

It holds that $p \sim_n q$ iff $s_i^p = s_i^q$ for all $i = 0, \dots, n + 1$.

So \mathbf{P}_L satisfies (C2) and (C3).

Theorem 2. Suppose that Axiom A poset P satisfies conditions (C1), (C2) and (C3). If P satisfies the following condition (C4), then P is not σ -short.

(C4): If $p \in P$ and X is a pairwise incomparable subset of P , then for every n there is $q \leq_n p$ such that $r \not\leq q$ for all $r \in X$.

Proof. Suppose that P is σ -short.

Then, there exists a family $\{X_n\}$ which satisfy the conditions as in Theorem 1.

We define a fusion sequence $\{p_n\}_{n \in \omega}$ inductively as follows.

Put $p_0 = p$. Suppose that p_n is already defined.

There exists $q \leq_n p_n$ such that $r \not\leq q$ for all $r \in X_n$ by (C4).

Let p_{n+1} be such an element q . Then $\{p_n\}_{n \in \omega}$ is a fusion sequence, so that there exists a fusion p_ω of $\{p_n\}_{n \in \omega}$.

Since $\bigcup_{n \in \omega} X_n$ is a dense subset of \mathbb{P} , there exists $n \in \omega$ and $r \in X_n$ such that $r \leq p_\omega$.

On the other hand, since $p_\omega \leq p_{n+1}$, we have $r' \not\leq p_\omega$ for all $r' \in X_n$ by virtue of the definition of p_{n+1} . This contradicts that $r \in X_n$ and $r \leq p_\omega$.

(C4) follows from (C2) and the following (C4a).

(C4a): If $p \in P$ and X is a pairwise incomparable subset of P such that $\forall r, r' \in X [r \sim_0 r']$, then for every n there is $q \leq_n p$ such that $r \not\leq q$ for all $r \in X$.

We can show that many Axiom A posets satisfy (C4a) using the strong type of amalgamations.

Example: Mathias forcing: (\mathbb{P}_M, \leq) is not σ -short.

(\mathbb{P}_M, \leq) is defined as follows.

$$\mathbb{P}_M = \{(s, S) \mid s \in \omega^{<\omega \uparrow}, S \subset_{inf} \omega \setminus \max(s)\},$$

$$(s, S) \geq (t, T) \Leftrightarrow t \supseteq s, T \subseteq S \text{ and } \text{range}(t) \setminus \text{range}(s) \subseteq S,$$

$$(s, S) \geq_n (t, T) \text{ iff } (s, S) \geq (t, T), s = t \text{ and } [S]_{n+1} = [T]_{n+1},$$

where $[S]_k$ is a set of the first k elements of S ,

$$p^* = (s, [S]_{n+1} \cup \{k \in \omega \mid k > \max([S]_{n+1})\}),$$

$$(s, S) \sim_n (t, T) \text{ iff } s = t \text{ and } [S]_{n+1} = [T]_{n+1}.$$

\mathbb{P}_M satisfies (C1), (C2) and (C3).

Let $p = (s, S), r = (t, T)$ and $p \geq r$.

We denote $(s, T \cup [S]_{n+1})$ by $p|_n r$ (or simply $p|r$) and call it n -amalgamation of r into p .

Lemma. Suppose that $p = (s, S), r = (t, T)$ and $p \geq r$.

(1) $p \geq_n p|_n r \geq r$,

(2) If $p|_n r \geq (t, T')$ and $T \cap [S]_{n+1} = T' \cap [S]_{n+1}$, then $r \geq r'$.

Proof. (1): Since $S \supseteq [S]_{n+1}$ and $S \supseteq T$, we have $S \supseteq T \cup [S]_{n+1}$ and $[T \cup [S]_{n+1}]_{n+1} = [S]_{n+1}$.

(2): Suppose that $p|_n r \geq (t, T')$ and $T \cap [S]_{n+1} = T' \cap [S]_{n+1}$.

Since $T \cup [S]_{n+1} \supseteq T'$ and $T \cap [S]_{n+1} = T' \cap [S]_{n+1}$, we have $T \supseteq T'$. Hence $r \geq r'$.

Lemma. Let X be a pairwise incomparable subset of P_M with same stem. Then for every $(s, S) \in P_M$ and $n \in \omega$, there exists $S' \subsetneq S$ such that $[S]_{n+1} = [S']_{n+1}$ and for every $(t, T) \in X$, $(s, S') \not\leq (t, T)$.

Proof. Let $p = (s, S) \in P_M$, $n \in \omega$ and $\mathcal{P}([S]_{n+1}) = \{\tau_1, \dots, \tau_{2^{n+1}}\}$. If $\forall (t, T) \in X[(s, S) \not\leq (t, T)]$, then take any S' such that $S' \subsetneq S$ and $[S]_{n+1} = [S']_{n+1}$. So we assume that there exists $r = (t, T) \in X$ such that $r \leq p$. We construct a sequence $\{q_k\}_{0 \leq k \leq 2^{n+1}+1}$ inductively such that $q_{k+1} = (s, S_{k+1}) \leq_n q_k = (s, S_k)$ for all k . Put $S_0 = S$.

Suppose that we already have q_k .

(1): If there exists $(t, T) \in X$ such that $(t, T) \leq (s, S_k)$ and $T \cap [S]_{n+1} = \tau_k$. We pick such an element $r = (t, T)$ and $T' \subsetneq T$ such that $T' \cap [S]_{n+1} = T' \cap [S]_{n+1}$, and put $q_{k+1} = q_k|(t, T')$.

(2): Otherwise, put $q_{k+1} = q_k$.

Finally we put $q = q_{2^{n+1}+1}$.

By virtue of the definition, we have $q \leq_n p$. We shall show that $q \not\geq r$ for all $r \in X$.

Suppose that $q \geq r = (t, T)$ for some $(t, T) \in X$.

Put $\tau = T \cap [S]_{n+1}$. Then $\tau = \tau_k$ for some k .

Thus we have $q_k \geq q \geq r$ and $T \cap [S]_{n+1} = \tau_k$.

So, by the definition of the sequece $\{q_k\}$, we have

defined $q_{k+1} = q_k|(t, T')$ where $T' \subsetneq T^*$ such that

$T' \cap [S]_{n+1} = T^* \cap [S]_{n+1} = \tau_k$ and $(t, T^*) \leq (s, S_k)$

for some $(t, T^*) \in X$.

Since $q_k|(t, T') \geq q \geq (t, T)$ and $T \cap [S]_{n+1} = T' \cap [S]_{n+1}$, $(t, T') \geq (t, T)$ by above amalgamation lemma. Hence we have $(t, T^*) > (t, T') \geq (t, T)$ and $(t, T), (t, T^*) \in X$. This contradicts that X is a pairwise incomparable subset of \mathbf{P}_M .

In the same way, we can prove that other posets, eg. Sacks forcing, Laver forcing, etc., satisfy (C4).

How to generalize this proof?

We need a definition of the strong type of amalgamation $p|r$ as above.

To define such amalgamations in general, we introduce frame systems for fusion sets.

Frame Systems

Let $(\mathbf{P}, \leq, \{\leq_n\}_{n \in \omega})$ be a fusion poset and f be a map from $\mathbf{P} \times \omega$ to ω .

$$I_{p,n}^* = \{k \in \omega \mid 0 \leq k \leq f(\text{stem}_n(p), n)\}.$$

We say that $\{a_{p,n,k} \in \mathbf{P} \mid n \in \omega, p \in \mathbf{P}, 0 \leq k \leq f(p, n)\}$ is a frame system for \mathbf{P} if it satisfies the following conditions.

(FS1): $\forall n \in \omega \forall p, q \in \mathbf{P} [p \sim_n q \Rightarrow f(p, n) = f(q, n)]$

(FS2): $\forall n \in \omega \forall p \in \mathbf{P} [\{a_{p,n,k}\}_{0 \leq k \leq f(p,n)}$ is a partition of p
and $\{a_{p,n+1,j}\}_{0 \leq j \leq f(p,n+1)}$ is a refinement of
 $\{a_{p,n,k}\}_{0 \leq k \leq f(p,n)}$.]

(FS3): $\forall n \in \omega \forall p, q \in \mathbf{P} [p \geq_n q$
 $\Rightarrow \forall k \in [0, f(p, n)] [a_{p,n,k} \geq_0 a_{q,n,k}]]$

(FS4): $\forall p, r \in \mathbf{P} [p \geq r \Rightarrow \exists n \in \omega \exists k \in [0, f(p, n)] [a_{p,n,k} \geq_0 r]]$

$$\begin{aligned}
 \text{(FS5): } & \forall n \in \omega \forall p, r \in \mathbf{P} [p \geq r \Rightarrow \exists q \leq_n p [q \geq r \wedge \\
 & \quad \forall r' \in \mathbf{P} [q \geq r' \wedge r \sim_0 r' \wedge r \sim_{p,n+1} r' \Rightarrow r \geq r']]
 \end{aligned}$$

$$\begin{aligned}
 \text{(FS6): } & \forall n \in \omega \forall p, r \in P [p \geq r \\
 & \quad \Rightarrow \exists r' \in P [r > r' \wedge r \sim_{p,n+1} r']]
 \end{aligned}$$

$$\text{(FS7): } \forall n \in \omega \forall p, r \in P [a_{p,n,k} \geq 0 \ r \Rightarrow a_{p,n,k} \sim_{p,n+1} r]$$

where

$$\begin{aligned}
 & r \sim_{p,n+1} r' \\
 & \Leftrightarrow \forall k \in I_{p,n+1}^* [r \uparrow a_{stem_n(p),n+1,k} \Leftrightarrow r' \uparrow a_{stem_n(p),n+1,k}]
 \end{aligned}$$

Let $n \in \omega, p, r \in \mathbf{P}$ and $p \geq r$. Then by (FS5), we can find $q \leq_n p$ such that $q \geq r \wedge \forall r' \in \mathbf{P}[q \geq r' \wedge r \sim_0 r' \wedge r \sim_{p, n+1} r' \Rightarrow r \geq r']$. We denote such an element q by $p|_n r$ (or simply $p|r$) and call it the n -amalgamation of r into p .

Examples:

Sacks forcing: Let $f(p, n) = 2^n - 1$ and $B_n(p) = \{s_0, \dots, s_{2^n - 1}\}$ (where $s_0 \prec s_1 \cdots \prec s_{2^n - 1}$).

Put $a_{p,n,k} = p \upharpoonright s_k = \{t \in p \mid t \subseteq s_k \text{ or } s_k \subseteq t\}$.

Then $\{a_{p,n,k} \in \mathbf{P}_S \mid n \in \omega, p \in \mathbf{P}_S, 0 \leq k \leq f(p, n)\}$ **is a frame system for** \mathbf{P}_S .

Mathias forcing:

$(s, S) \geq_n (t, T)$ iff $(s, S) \geq (t, T)$, $s = t$ and $[S]_{n+1} = [T]_{n+1}$.

Let $f(p, n) = 2^n - 1$, $m = \max([S]_{n+1})$ and

$K_n = \{\tau \in \omega^{<\omega} \uparrow \mid \mathbf{range}(\tau) \subseteq [S]_n\} = \{\tau_0, \dots, \tau_{2^n-1}\}$

(where $\tau_0 \prec \tau_1 \cdots \prec \tau_{2^n-1}$).

For $p = (s, S)$, put $a_{p,n,k} = (s \hat{\ } \tau_k, S \setminus [S]_n)$.

$stem_n(p) = (s, [S]_{n+1} \cup \{k \in \omega \mid k > m\})$

$$\begin{aligned}
& \{a_{stem_n(p),n+1,j} \mid j \in I_{p,n+1}^*\} \\
& = \{(s \hat{\ } \tau_k, \{k \in \omega \mid k > m\}) \mid \tau_k \in K_n\} \cup \\
& \quad \{(s \hat{\ } \tau_k \hat{\ } \langle m \rangle, \{k \in \omega \mid k > m\}) \mid \tau_k \in K_n\}
\end{aligned}$$

Lemma.

(1) If $(s, S) \geq_n q = (s, S')$, then $a_{p,n,k} \geq_0 a_{q,n,k}$.

(2) If $p = (s, S) \geq (t, T), (t, T')$, then

$$(t, T) \sim_{p,n+1} (t, T') \Leftrightarrow T \cap [S]_{n+1} = T' \cap [S]_{n+1}.$$

(3) If $a_{n,p,k} \geq_0 (t, T)$, then $a_{n,p,k} \sim_{p,n+1} (t, T)$.

So, $\{a_{p,n,k} \in \mathbf{P}_M \mid n \in \omega, p \in \mathbf{P}_M, 0 \leq k \leq f(p, n)\}$ is a frame system for \mathbf{P}_M .

Laver forcing:

$p \geq_n q$ iff $p \geq q$ and $s_i^p = s_i^q$ for all $i = 0, \dots, n + 1$.

Let $f(p, n) = n$ and $K_n = \mathcal{P}(\{s_0^p, \dots, s_n^p\})$.

For $p \in P_L$, $a_{p,n,k} = (p \upharpoonright s_k^p) \setminus \{t \in p \mid \exists j > k [s_j^p \subseteq t]\}$.

If s_k^p is \subseteq -maximal node among K_n , then $a_{p,n,k} = p \upharpoonright s_k^p$.

$stem_n(p) = \{t \in p \mid t \subseteq s_0^p\} \cup \{s_1^p, \dots, s_{n+1}^p\} \cup \{t \in \omega^{<\omega} \mid s_{n+1}^p \prec t\}$.

Let $k = \max\{j \mid s_j^p \subsetneq s_{n+1}^p\}$. Then, if $k \neq j \leq n$, then

$$a_{stem_n(p), n+1, j} = a_{stem_n(p), n, j}.$$

$$a_{stem_n(p), n+1, k} = a_{stem_n(p), n, k} \setminus \{t \in stem_n(p) \mid s_{n+1}^p \subseteq t\}.$$

Lemma.

(1) If $p \geq_n q$, then $a_{p,n,k} \geq_0 a_{q,n,k}$.

(2) If $p \geq r, r'$ and $r \sim_0 r'$, then

$$r \sim_{p,n+1} r' \Leftrightarrow r \cap K_{n+1} = r' \cap K_{n+1}.$$

(3) If $a_{n,p,k} \geq_0 r$, then $a_{n,p,k} \sim_{p,n+1} (t, T)$.

Lemma (Amalgamation Lemma). Let $(\mathbb{P}, \leq, \{\leq_n\}_{n \in \omega})$ be a fusion poset with a frame system which satisfies (C1), (C2) and (C3). If $n \in \omega, p, r \in \mathbb{P}$ and $r \leq_0 a_{p,n,k}$, then we have $a_{p|r,n,k} = r$.

Proof. Suppose that $n \in \omega, p, r \in \mathbb{P}$ and $r \leq_0 a_{p,n,k}$. Let $q = p|r$. Then we have

$$(*) \quad \forall r' \in \mathbb{P} [q \geq r' \wedge r \sim_0 r' \wedge r \sim_{p,n+1} r' \Rightarrow r \geq r']$$

Since $\{a_{p,n,k}\}_{0 \leq k \leq f(p,n)}$ is a partition of p and $r \leq a_{p,n,k}$, r is not compatible with $a_{p,n,j}$ for all $j \neq k$. By virtue

of (FS3), we have $a_{p,n,j} \geq a_{q,n,j}$. So r is not compatible with $a_{q,n,j}$ for all $j \neq k$.

Since $q = p|_r \geq r$, we have $a_{q,n,k} \geq r$.

We have $a_{p,n,k} \geq_0 a_{q,n,k}$ by (FS3), so that we have $r \sim_0 a_{q,n,k}$. Hence by virtue of (CS3), $a_{q,n,k} \geq_0 r$. Therefore $a_{q,n,k} \sim_{q,n+1} r$ by (FS7). Since $p \geq_n q$, we have $stem_n(p) = stem_n(q)$, so that $a_{q,n,k} \sim_{p,n+1} r$. So we have $r \geq a_{q,n,k}$ by (*). Thus $a_{q,n,k} = a_{p|_r,n,k} = r$.

Lemma. Let $(P, \leq, \{\leq_n\}_{n \in \omega})$ be a fusion poset with a frame system which satisfies (C1), (C2) and (C3). Suppose that W is a partition of P and $p \in P$. Then there exists $q \leq_0 p$ such that q is compatible with at most countably many $r \in W$.

Proof. It follows from the Amalgamation Lemma by usual arguments.

Theorem. If $(P, \leq, \{\leq_n\}_{n \in \omega})$ is a fusion poset with a frame system which satisfies (C1), (C2) and (C3), then $(P, \leq, \{\leq_n\}_{n \in \omega})$ satisfies (A4).

Proof. By usual arguments.

Teorem. If $(P, \leq, \{\leq_n\}_{n \in \omega})$ is a fusion poset with a frame system which satisfies (C1), (C2) and (C3), then $(P, \leq, \{\leq_n\}_{n \in \omega})$ satisfies (C4).

Proof. Let $\{a_{p,n,k}\}$ be a frame system for P , X be a pairwise incomparable subset of P such that $\forall r, r' \in X [r \sim_0 r']$ and $p \in P$.

We shall show that there exists $q \leq_n p$ such that $r \not\leq q$ for all $r \in X$.

If there exists no $r \in X$ such that $r \leq p$, then we put $q = p$.

So we assume that there exists $r \in X$ such that $r \leq p$.

Let $\ell = f(\text{stem}_n(p), n + 1)$ and $\mathcal{P}(I_{p, n+1}^*) = \{t_1, \dots, t_{2^{\ell+1}}\}$.

We construct a sequence $\{q_k\}_{0 \leq k \leq 2^{\ell+1} + 1}$ inductively such that

$q_{k+1} \leq_n q_k$ for all k . Put $q_0 = p$.

Suppose that we already have q_k . In the following, we denote $\{j \mid r \uparrow a_{\text{stem}_n(p), n+1, j}\}$ by $C(r)$.

(1): If there exists $r \in X$ such that $r \leq q_k$ and $C(r) = t_k$.

We pick such an element r and take $\tilde{r} < r$ such that

$r \sim_{p, n+1} \tilde{r}$ by (FS6). Then put $q_{k+1} = q_k | \tilde{r}$.

(2): Othewise, put $q_{k+1} = q_k$.

Finally we put $q = q_{2^{\ell+1}+1}$.

By virtue of the definition, we have $q \leq_n p$. So we shall show that $q \not\geq r$ for all $r \in X$. Suppose that $q \geq r$ for some $r \in X$. Put $t = C(r)$. Then $t = t_k$ for some k . Thus we have $q_k \geq q \geq r$ and $C(r) = t_k$. So, by the definition of the sequece $\{p_k\}$, we have defined $q_{k+1} = q_k | \tilde{r}$ where $\tilde{r} < r^*$, $\tilde{r} \sim_{p, n+1} r^*$ and $C(r^*) = t_k$ for some $r^* \in X$. Then $C(\tilde{r}) = C(r^*) = t_k = C(r)$. Since $q_k | \tilde{r} = q_{k+1} \geq q \geq r$, $\tilde{r} \geq r$ by (FS5). Hence we have $r^* > \tilde{r} \geq r$ and $r^*, r \in X$. This contradicts that X is a pairwise incomparable subset of P .

Theorem. Suppose that $(P, \leq, \{\leq_n\}_{n \in \omega})$ is a fusion poset with a frame system which satisfies (C1), (C2) and (C3). Then, $(P, \leq, \{\leq_n\}_{n \in \omega})$ is not σ -short.

Finiteness Property

H. Mildenberger, The club principle and the distributivity number, Journal of Symbolic Logic, Vol. 76 No.1,2011, pp. 34-46

In this paper, Mildenberger defined the finiteness property for Axiom A posets. It is defined as follows.

Definition. An Axiom A poset $(P, \leq, \{\leq_n\}_{n \in \omega})$ whose elements are subsets of $2^{<\omega}$ or of $\omega^{<\omega}$ has the finiteness property iff

1. $p \geq q$ implies $p \supseteq q$,

2. there is a function $h : P \times \omega \longrightarrow \omega$ such that for every n, p, q ,

$$p \geq_n q \text{ iff } p \geq q \text{ and } q \cap h(p, n)^{h(p, n)} = p \cap h(p, n)^{h(p, n)}.$$

In the case of $2^{<\omega}$, we can write $2^{h(p, n)}$ instead of

$h(p, n)^{h(p, n)}$. We denote $2^{h(p, n)}$ or $h(p, n)^{h(p, n)}$ by H_n^p .

Without loss of generality, we may assume that elements of \mathbb{P} are trees. We say that \mathbb{P} has the **uniform finiteness property** if it has the finiteness property and **for every** $n \in \omega, p, q \in \mathbb{P}, p \geq_n q$ **implies** $h(p, n) = h(q, n)$.

For $p \in \mathbb{P}$, $s \in p$ is called **the stem of p** if

(i): for every $t \in p$, $s \subseteq t$ or $t \subseteq s$, and

(ii): p is a branching point, i.e., s has at least two successors in p .

We denote the stem of p as $st(p)$. If σ is a finite subtree of p , we denote it by $\sigma \in p$.

We say that $t \in \sigma$ is a σ -branching point of p if there exists $k \in \omega$ such that $t \frown \langle k \rangle \in p$ and $t \frown \langle k \rangle \notin \sigma$. We denote the set of σ -branching points of p by σ^b .

If $\mathbb{P} = P_L$, then $\sigma^b = \sigma \setminus \{t \in \sigma \mid t \subsetneq st(p)\}$

Let $p \geq r$ and $t \in \sigma^b$. Then we say that t is a r - σ -branching point of p if there exists $s \in r$ such that $t \subsetneq s$ and $\forall k \in \omega [t \frown \langle k \rangle \subseteq s \Rightarrow t \frown \langle k \rangle \notin \sigma]$. We denote the set of r - σ -branching points of p by $\sigma^{b,r}$. For $p \geq r, r'$ and $\sigma \in p$, we define $r \approx_\sigma r'$ if and only if $r \cap \sigma = r' \cap \sigma$ and $\sigma^{b,r} = \sigma^{b,r'}$.

We say that \mathbf{P} has enough elements if \mathbf{P} satisfies the following

1. $I = 2^{<\omega}$ or $\omega^{<\omega} \in \mathbf{P}$,

2. for every $r \in \mathbf{P}$, there exists $r' \in \mathbf{P}$ such that $r > r'$ and $st(r) = st(r')$,

3. for every $p \in \mathbf{P}$,

$$p^* = I \setminus \{t \in I \mid t \not\subseteq p, \exists s \in (H_n^p \setminus p) [s \subseteq t \text{ or } t \subseteq s]\} \in \mathbf{P},$$

4. for every $p \in \mathbf{P}$ and $s \in p$,

$$p \upharpoonright s = \{t \in p \mid t \subseteq s \text{ or } s \subseteq t\} \in \mathbf{P},$$

5. for every $p \in \mathbf{P}$ and $r \leq p$,

$$p \upharpoonright r = r \cup \{t \in p \mid t \not\subseteq st(r) \text{ and } st(r) \not\subseteq t\} \in \mathbf{P}.$$

Let $\sigma_p^n = \{t \in \omega^{<\omega} \mid \exists s \in p \cap H_n^p [t \subseteq s]\}$.

Lemma. Let $(P, \leq, \{\leq_n\}_{n \in \omega})$ be an Axiom A poset with uniform finiteness property which has enough elements. Then for every $n \in \omega, p \in P$ and $p \geq r$, there exists $r' < r$ such that $r \approx_{\sigma_p^n} r'$.

Theorem. Let $(P, \leq, \{\leq_n\}_{n \in \omega})$ be an **Axiom A** poset with **uniform finiteness property** which has enough elements.

Then we have

1. P satisfies **(C1)**, **(C2)** and **(C3)**.
2. If $(P, \leq, \{\leq_n\}_{n \in \omega})$ satisfies the following **strong amalgamation property**, then P is not σ -short.

$(AP) : \forall n \in \omega \forall p \in P \forall r \in P [p \geq r \Rightarrow$

$$\exists q \leq_n p \left[q \geq r \wedge \forall r' \in P [q \geq r' \wedge r \approx_{\sigma_p^n} r' \Rightarrow r \geq r'] \right]$$

In the following, we assume that

(FS8): $\forall p \in \mathbf{P} \forall n \in \omega \forall k \in I_{p,n}^*$

$$\exists m \in \omega \exists j \in I_{1,m} [a_{stem_n(p),n,k} = a_{1,m,j}]$$

(FS9): $\forall p, r, r' \in \mathbf{P} [p \geq r, r' \Rightarrow [\forall n \in \omega [r \sim_{p,n} r'] \Rightarrow r = r']]$.

(FS10): $\forall p \in \mathbf{P} \exists q \leq p [q \text{ is uniform}]$,

where q is uniform if for every $n \in \omega$,

$$\max\{m \mid \exists k, j [a_{stem_n(p),n,k} = a_{1,m,j}]\}$$

$$< \min\{m \mid \exists k, j [a_{stem_{n+1}(p),n+1,k} = a_{1,m,j}]\}.$$

For every $p \in \mathbf{P}$, we define a subtree \tilde{p} of $\omega^{<\omega}$ by

$$\tilde{p} = \{\tau \in \omega^{<\omega} \mid \forall n \in \text{dom}(\tau) [0 \leq \tau(n) \leq f(1, n), p \uparrow a_{1, n, \tau(n)}]\}.$$

Let $\tilde{\mathbf{P}} = \{\tilde{p} \mid p \in \mathbf{P}\}$. We define a partial order \leq_T on

$\tilde{\mathbf{P}}$ such that $\tilde{p} \leq_T \tilde{q}$ if and only if \tilde{p} is a subtree of \tilde{q} .

We denote $I_{1, n}$ by I_n . For $n \in \omega$ and $j \in I_n$, we define

$\tau_j^n \in \omega^{<\omega}$ by $\text{dom}(\tau_j^n) = \{0, \dots, n\}$ and $\tau_j^n(k) = \ell$ if and only

if $a_{1, k, \ell} \geq a_{1, n, j}$. Since $\{a_{1, k, \ell}\}_{0 \leq \ell \leq f(1, k)}$ is a partition of 1

and $\{a_{1, n, j}\}_{0 \leq j \leq f(1, n)}$ is a refinement of $\{a_{1, k, \ell}\}_{0 \leq \ell \leq f(1, k)}$ for

$k \leq n$, τ_j^n is well-defined. If $a_{\text{stem}_n(p), n, k} = a_{1, m, j}$, we de-

note τ_j^m by $\tau_{p, n, k}$.

Lemma. \tilde{p} is a subtree of $\omega^{<\omega}$ for every $p \in P$.

Lemma. For every $\tau \in \tilde{p}$, there are extensions τ_1, τ_2 of τ such that τ_1 and τ_2 are incompatible.

Lemma.

1. If $p \neq q$, then $\tilde{p} \neq \tilde{q}$.
2. If $p \leq q$, then \tilde{p} is a subtree of \tilde{q} .
3. If $p \perp q$, then $\tilde{p} \perp \tilde{q}$.

Lemma. If (P, \leq) is a fusion poset with a frame system which satisfies (F8) and (F9), then (P, \leq) is isomorphic to (\tilde{P}, \leq_T) .

Let $\widetilde{P}_u = \{\tilde{p} \mid p \text{ is uniform}\}$. Then \widetilde{P}_u is a dense subset of \widetilde{P} by (F10).

Theorem. If $\forall n \forall p \forall k [|\{j \mid a_{p,n,k} > a_{p,n+1,j}\}| = 2]$, then $(\widetilde{P}_u, \leq_T)$ satisfies the finiteness property.

It is open that $(\widetilde{P}_u, \leq_T)$ satisfies the finiteness property, in general.

Remark. Since $\widetilde{P}_L \cong P_L$, \widetilde{P}_L satisfies the finiteness property as in [2].

Open Problems

1. Hechler forcing which adds a strictly increasing function from ω_1 to ω_1 is not σ -short. How about general Hechler forcing?
2. Is a forcing product of a σ -closed poset and a CCC poset with the density $\geq \omega_1$ not σ -short?
3. Is Axiom A non-CCC poset not σ -short?

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