On non σ-shortness of Axiom A posets with frame systems

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Previous Study

In this talk, we assume that Boolean algebras are atomless, posets are nom-atomic and separative.

Definition

$D \subseteq \mathrm{B}^+$ is *σ-short* if every strictly descending sequence of length ω in D does not have a nonzero lower bound in B.

B is said to be *σ-short* if it has a σ-short dense subset.

B is said to be *strongly σ-short* if it has a σ-short $\wedge$-closed dense subset.

D is $\wedge$-*closed* if and only if $x \wedge y \in D$ for every $x, y \in D$.

We note that B itself is not a σ-short set, since B is atomless.

σ-short posets are defined the same way as for BAs.

Examles of σ-short Boolean algebras

1. For any set X, let $\mathsf{Fr}\,X$ be the free Boolean algebra over X.

$$D = \{\pm x_1 \cdot \pm x_2 \cdot \ldots \cdot \pm x_n \,\big|\, n \in \omega, x_1, x_2, \cdots, x_n \in X\} - \{0\}$$

Clearly, D is a σ-short $\wedge$-closed dense subset of $\mathsf{Fr}\,X$. Hence, $\mathsf{Fr}\,X$ is strongly σ-short.

Every regularly filtered Ba is also strongly σ-short[TY].

2. Let (B, μ) be a measure algebra.

$D = \{a \in \mathrm{B} \,\big|\, \mu(a) = \frac{1}{n+1}$ **for some** $n \in \omega\}$.

D is a σ-short dense subset of B. Hence (B, μ) is σ-short.

Characterizations of σ-short BAs:

Q1. Is it true that the following are equivalent?

(1) B is σ-short.

(2) There exists a sequence $\{X_n\}_{n\in\omega}$ of subsets of B which satisfies the following conditions:

(a) X_n is a pairwise incomparable subset of B.

(b) If $x \in X_n, y \in X_m$ and $n < m$, then $y \not\geq x$.

(c) $X = \bigcup\limits_{n\in\omega} X_n$ is a dense subset of B.

It holds that (2)$\Rightarrow$(1).

Partial answer for strongly σ-short BAs

Theorem.([T]) The following are equivalent.

(1) B is strongly σ-short.

(2) There exists a sequence $\{X_n\}_{n\in\omega}$ of subsets of B which satisfies the following conditions:

(a) X_n is a pairwise incomparable subset of B.

(b) If $x \in X_n, y \in X_m$ and $n < m$, then $y \not\geq x$.

(c) $X = \bigcup_{n\in\omega} X_n$ is a dense subset of B

(d) $\{y \in X_n | y \geq x\}$ is finite for every $n \in \omega$ and $x \in X_{n+1}$.

Q.2 Is it true that every σ-short BAs are strongly σ-short?

No.

Theorem A(Brendle). Let B_κ be the algebra for adding κ many random reals.
(1) B_ω is not strongly σ-short.
(2) Assume that the density of B_κ equals to κ. Then B_κ is strongly σ-short.

Partial answer to Q.1 for Axiom A posets:

Theorem 1. Let P be σ-short Axiom A poset. If P satisfies the conditions (C1), (C2) and (C3) (to be defined later), then there exists a sequence $\{X_n\}_{n\in\omega}$ of subsets of P which satisfies the the following conditions:

(a) X_n is a pairwise incomparable subset of B.

(b) If $x \in X_n, y \in X_m$ and $n < m$, then $y \not\geq x$.

(c) $X = \bigcup_{n\in\omega} X_n$ is a dense subset of B.

Actually, many Axiom A posets are not σ-short.

A poset $(\mathrm{P}, \leq)$ satisfies Axiom A if there are partial orderings $\leq_n$ $(n \in \omega)$ such that

(A1): If $p \leq_0 q$ then $p \leq q$;

(A2): If $p \leq_{n+1} q$, then $p \leq_n q$;

(A3): If $\{p_n\}_{n\in\omega}$ is a fusion sequence; i.e., if $p_{n+1} \leq_n p_n$ for every $n \in \omega$, then there is q such that $q \leq_n p_n$ for all $n \in \omega$;

(A4): If $p \in \mathrm{P}$ and W is a partition of p, then for every n there is $q \leq_n p$ such that q is compatible with at most countably many $x \in W$.

We say that a poset $(\mathrm{P}, \leq)$ with partial orderings $\leq_n$ $(n \in \omega)$ is a *fusion poset* if it satisfies (A1),(A2),(A3). We assume that partial orderings $\{\leq_n\}_{n\in\omega}$ are transitive.

Fir st, we consider the following condition (C1) for fusion posets.

(C1):$\forall n \in \omega \forall p \in \mathrm{P} \exists p^* \geq_n p \forall p' \geq_n p [p^* \geq_n p']$

For $n \in \omega$ and $p \in \mathrm{P}$, we denote $*p$ in (C1) by $stem_n(p)$.

If a fusion poset $\mathbb{P}$ satisfies (C1), then the relation $\sim_n$ on $\mathbb{P}$ defined by $p \sim_n q \overset{\textit{def}}{\iff} stem_n(p) = stem_n(q)$ is an equivalence relation on $\mathbb{P}$.

Using this equivalence relation, we consider conditions (C2) and (C3) as follows.

(C2): $\forall n \in \omega[|\mathbb{P}/\sim_n| \leq \omega]$

(C3): $\forall n \in \omega \forall p, q \in \mathbb{P}[p \sim_n q \ \& \ p \geq q \Rightarrow p \geq_n q]$

Examples

In the following examples, we consider a canonical enumeration of $2^{<\omega}$ or $\omega^{<\omega}$. And, when we enumerate elements of a subset of those sets, we use this canonical enumeration.

If t appears in an enumeration after s, then we denote it by $s \prec t$.

Let $\omega^{<\omega}\uparrow = \{t \in \omega^{<\omega} \mid t$ is increasing$\}$.

Sacks forcing: $(\mathrm{P}_S, \leq)$ **is defined as follows.**

$\mathrm{P}_S = \{p \mid p$ **is a perfect tree of** $2^{<\omega}\}$ **and** $p \geq q$ **iff** $p \supseteq q$**.**

$p \geq_n q \Leftrightarrow p \geq q$ *and* $B_n(p) = B_n(q)$ **where** $B_n(p)$ **is a set of the** $(n+1)$**-st branching points of** p**.**

For $p \in \mathrm{P}_S$ **and** $n \in \omega$**, put**

$stem_n(p) = \{t \in 2^{<\omega}\} \mid \exists s \in B_n(p)\,[t \subseteq s$ *or* $s \subseteq t]\}$.

P_S **satisfies (C1). It holds that** $p \sim_n q$ **iff** $B_n(p) = B_n(q)$**. So** P_S **satisfies (C2) and (C3).**

Laver forcing: $(\mathrm{P}_L, \leq)$ **is defined as follows.**

$\mathrm{P}_L = \{p \mid$ p **is a tree of** $\omega^{<\omega}$ **which has a stem** s **such that** $\forall t \supseteq s[S(t) = \{k \in \omega \mid t^\frown k \in p\}$ **is infinite**$]\}$ **and** $p \geq q \Leftrightarrow p \supseteq q$.

For $p \in \mathrm{P}_L$, **let** $s_0^p = \mathrm{stem}(p), s_1^p, \ldots, s_m^p, \ldots$ **be an enumeration of** $\{t \in p \mid t \supseteq \mathrm{stem}(p)\}$.

$p \geq_n q$ **iff** $p \geq q$ **and** $s_i^p = s_i^q$ **for all** $i = 0, \ldots, n+1$.

For $p \in \mathrm{P}_L$ **and** $n \in \omega$**,**

$stem_n(p) = \{t \in p \mid t \subseteq s_0^p\} \cup \{s_1^p, \ldots, s_{n+1}^p\} \cup \{t \in \omega^{<\omega} \mid s_{n+1}^p \prec t\}$**.**

P_L **satisfies (C1).**

It holds that $p \sim_n q$ **iff** $s_i^p = s_i^q$ **for all** $i = 0, \ldots, n+1$**.**

So P_L **satisfies (C2) and (C3).**

Theorem 2. Suppose that Axiom A poset P satisfies conditions $(\mathrm{C1})$,$(\mathrm{C2})$ and $(\mathrm{C3})$. If P satisfies the following condition $(\mathrm{C4})$, then P is not σ-short.

(C4):If $p \in \mathrm{P}$ and X is a pairwise incomparable subset of P, then for every n there is $q \leq_n p$ such that $r \nleq q$ for all $r \in X$.

Proof. Suppose that P is σ-short.

Then, there exists a family $\{X_n\}$ which satisfy the conditions as in Theorem 1.

We define a fusion sequence $\{p_n\}_{n\in\omega}$ inductively as follows.

Put $p_0 = p$. Suppose that p_n is already defined.

There exists $q \leq_n p_n$ such that $r \not\leq q$ forall $r \in X_n$ by (C4). Let p_{n+1} be such an element q. Then $\{p_n\}_{n\in\omega}$ is a fusion sequence, so that there exists a fusion p_ω of $\{p_n\}_{n\in\omega}$.

Since $\bigcup_{n\in\omega} X_n$ is a dense subset of $\mathbb{P}$, there exists $n \in \omega$ and $r \in X_n$ such that $r \leq p_\omega$.

On the other hand, since $p_\omega \leq p_{n+1}$, we have $r' \not\leq p_\omega$ for all $r' \in X_n$ by virtue of the definition of p_{n+1}. This contradicts that $r \in X_n$ and $r \leq p_\omega$.

(C4) follows from (C2) and the following (C4a).

(C4a): If $p \in \mathrm{P}$ and X is a pairwise incomparable subset of P such that $\forall r, r' \in X \left[r \sim_0 r' \right]$, then for every n there is $q \leq_n p$ such that $r \not\leq q$ for all $r \in X$.

We can show that many Axiom A posets satisfy (C4a) using the strong type of amalgamations.

Example: Mathias forcing: $(\mathrm{P}_M, \leq)$ is not σ-short.

$(\mathrm{P}_M, \leq)$ **is defined as follows.**

$\mathrm{P}_M = \{(s, S) \ \mid \ s \in \omega^{<\omega}\uparrow, S \subset_{inf} \omega \setminus \max(s)\}$,

$(s, S) \geq (t, T) \Leftrightarrow t \supseteq s, T \subseteq S$ **and** $\mathrm{range}(t) \setminus \mathrm{range}(s) \subseteq S$,

$(s, S) \geq_n (t, T)$ *iff* $(s, S) \geq (t, T), s = t$ *and* $[S]_{n+1} = [T]_{n+1}$,
where $[S]_k$ **is a set of the first** k **elements of** S,

$p^* = (s, [S]_{n+1} \cup \{k \in \omega \mid k > \max([S]_{n+1})\})$,

$(s, S) \sim_n (t, T)$ **iff** $s = t$ **and** $[S]_{n+1} = [T]_{n+1}$.

P_M **satisfies (C1),(C2) and (C3).**

Let $p = (s, S), r = (t, T)$ and $p \geq r$.

We denote $(s, T \cup [S]_{n+1})$ by $p|^n r$(or simply $p|r$) and call it n-amalgamation of r into p.

Lemma. Suppose that $p = (s, S), r = (t, T)$ and $p \geq r$.

(1) $p \geq_n p|^n r \geq r$,

(2) If $p|^n r \geq (t, T')$ and $T \cap [S]_{n+1} = T' \cap [S]_{n+1}$, then $r \geq r'$.

Proof. (1): Since $S \supseteq [S]_{n+1}$ and $S \supseteq T$, we have $S \supseteq T \cup [S]_{n+1}$ and $[T \cup [S]_{n+1}]_{n+1} = [S]_{n+1}$.

(2): Suppose that $p|^n r \geq (t, T')$ and $T \cap [S]_{n+1} = T' \cap [S]_{n+1}$.

Since $T \cup [S]_{n+1} \supseteq T'$ and $T \cap [S]_{n+1} = T' \cap [S]_{n+1}$, we have $T \supseteq T'$. Hence $r \geq r'$.

Lemma. Let X be a pairwise incomparable subset of P_M with same stem. Then for every $(s, S) \in P_M$ and $n \in \omega$, there exists $S' \subsetneq S$ such that $[S]_{n+1} = [S']_{n+1}$ and for every $(t, T) \in X$, $(s, S') \not\geq (t, T)$.

Proof. Let $p = (s, S) \in P_M$, $n \in \omega$ and $\mathcal{P}([S]_{n+1}) = \{\tau_1, \ldots, \tau_{2^{n+1}}\}$. If $\forall(t, T) \in X[(s, S) \not\geq (t, T)]$, then take any S' such that $S' \subsetneq S$ and $[S]_{n+1} = [S']_{n+1}$. So we assume that there exists $r = (t, T) \in X$ such that $r \leq p$. We construct a sequence $\{q_k\}_{0 \leq k \leq 2^{n+1}+1}$ inductively such that

$q_{k+1} = (s, S_{k+1}) \leq_n q_k = (s, S_k)$ for all k. Put $S_0 = S$.

Suppose that we already have q_k.

(1): If there exists $(t, T) \in X$ such that $(t, T) \leq (s, S_k)$ and $T \cap [S]_{n+1} = \tau_k$. We pick such an element $r = (t, T)$ and $T' \subsetneq T$ such that $T' \cap [S]_{n+1} = T' \cap [S]_{n+1}$, and put $q_{k+1} = q_k|(t, T')$.

(2): Otherwise, put $q_{k+1} = q_k$.

Finally we put $q = q_{2^{n+1}+1}$.

By virtue of the definition, we have $q \leq_n p$. We shall show that $q \not\geq r$ for all $r \in X$.

Suppose that $q \geq r = (t, T)$ for some $(t, T) \in X$.

Put $\tau = T \cap [S]_{n+1}$. Then $\tau = \tau_k$ for some k.

Thus we have $q_k \geq q \geq r$ and $T \cap [S]_{n+1} = \tau_k$.

So, by the definition of the sequece $\{q_k\}$, we have

defined $q_{k+1} = q_k|(t, T')$ where $T' \subsetneq T^*$ such that

$T' \cap [S]_{n+1} = T^* \cap [S]_{n+1} = \tau_k$ and $(t, T^*) \leq (s, S_k)$

for some $(t, T^*) \in X$.

Since $q_k|(t, T') \geq q \geq (t, T)$ and $T \cap [S]_{n+1} = T' \cap [S]_{n+1}$, $(t, T') \geq (t, T)$ by above amalgamation lemma. Hence we have $(t, T^*) > (t, T') \geq (t, T)$ and $(t, T), (t, T^*) \in X$. This contradicts that X is a pairwise incomparable subset of P_M.

In the same way, we can prove that other posets, eg. Sacks forcing, Laver forcing,etc., satisfy (C4).

How to generalize this proof?

We need a definition of the strong type of amalgamation $p|r$ as above.

To define such amalgamations in general, we introduce frame systems for fusion sets.

Frame Systems

Let $(\mathrm{P}, \leq, \{\leq_n\}_{n\in\omega})$ be a fusion poset and f be a map from $\mathrm{P} \times \omega$ to ω.

$I^*_{p,n} = \{k \in \omega \mid 0 \leq k \leq f(stem_n(p), n)\}$.

We say that $\{a_{p,n,k} \in \mathrm{P} \mid n \in \omega, p \in \mathrm{P}, 0 \leq k \leq f(p, n)\}$ is a frame system for P if it satisfies the following conditions.

(FS1): $\forall n \in \omega \forall p, q \in \mathbf{P}[p \sim_n q \Rightarrow f(p,n) = f(q,n)]$

(FS2): $\forall n \in \omega \forall p \in \mathbf{P} \left[\{a_{p,n,k}\}_{0 \le k \le f(p,n)} \right.$**is a partition of** p

and $\{a_{p,n+1,j}\}_{0 \le j \le f(p,n+1)}$**is a refinement of**

$\left. \{a_{p,n,k}\}_{0 \le k \le f(p,n)}. \right]$

(FS3): $\forall n \in \omega \forall p, q \in \mathbf{P} [p \ge_n q$

$$\Rightarrow \forall k \in [0, f(p,n)] \left[a_{p,n,k} \ge_0 a_{q,n,k} \right]]$$

(FS4): $\forall p, r \in \mathbf{P} \left[p \ge r \Rightarrow \exists n \in \omega \exists k \in [0, f(p,n)] \left[a_{p,n,k} \ge_0 r \right] \right]$

(FS5): $\forall n \in \omega \forall p, r \in \mathbf{P}\,[p \geq r \Rightarrow \exists q \leq_n p\,[q \geq r \wedge$

$$\forall r' \in \mathbf{P}[q \geq r' \wedge r \sim_0 r' \wedge r \sim_{p,n+1} r' \Rightarrow r \geq r']]$$

(FS6): $\forall n \in \omega \forall p, r \in P\,[p \geq r$

$$\Rightarrow \exists r' \in P\left[r > r' \wedge r \sim_{p,n+1} r'\right]]$$

(FS7): $\forall n \in \omega \forall p, r \in P\left[a_{p,n,k} \geq_0 r \Rightarrow a_{p,n,k} \sim_{p,n+1} r\right]$

where

$r \sim_{p,n+1} r'$

$$\Leftrightarrow \forall k \in I^*_{p,n+1}\left[r \uparrow a_{stem_n(p),n+1,k} \Leftrightarrow r' \uparrow a_{stem_n(p),n+1,k}\right]$$

Let $n \in \omega, p, r \in \mathrm{P}$ and $p \geq r$. Then by (FS5), we can find $q \leq_n p$ such that $q \geq r \wedge \forall r' \in \mathrm{P}[q \geq r' \wedge r \sim_0 r' \wedge r \sim_{p,n+1} r' \Rightarrow r \geq r']$. We denote such an element q by $p|^n r$(or simply $p|r$) and calll it the n-amalgamation of r into p.

Examples:

Sacks forcing: Let $f(p,n) = 2^n - 1$ **and** $B_n(p) = \{s_0, \ldots, s_{2^n-1}\}$ **(where** $s_0 \prec s_1 \cdots \prec s_{2^n-1}$**).**

Put $a_{p,n,k} = p{\restriction}s_k = \{t \in p \mid t \subseteq s_k$ **or** $s_k \subseteq t\}$**.**

Then $\{a_{p,n,k} \in \mathrm{P}_S \mid n \in \omega, p \in \mathrm{P}_S, 0 \le k \le f(p,n)\}$ **is a frame system for** P_S**.**

Mathias forcing:

$(s,S) \geq_n (t,T)$ *iff* $(s,S) \geq (t,T), s = t$ *and* $[S]_{n+1} = [T]_{n+1}$.

Let $f(p,n) = 2^n - 1$, $m = \max([S]_{n+1})$ **and**

$K_n = \{\tau \in \omega^{<\omega}\!\uparrow \mid \textbf{range}(\tau) \subseteq [S]_n\} = \{\tau_0, \ldots, \tau_{2^n-1}\}$

(where $\tau_0 \prec \tau_1 \cdots \prec \tau_{2^n-1}$**).**

For $p = (s,S)$, **put** $a_{p,n,k} = (s^\frown\tau_k, S\backslash[S]_n)$.

$stem_n(p) = (s, [S]_{n+1} \cup \{k \in \omega \mid k > m)\})$

$$\{a_{stem_n(p),n+1,j} \mid j \in I^*_{p,n+1}\}$$

$$= \{(s^\frown\tau_k, \{k \in \omega \mid k > m)\}) \mid \tau_k \in K_n\} \cup$$

$$\{(s^\frown\tau_k^\frown\langle m\rangle, \{k \in \omega \mid k > m)\}) \mid \tau_k \in K_n\}$$

Lemma.

(1) If $(s, S) \geq_n q = (s, S')$**, then** $a_{p,n,k} \geq_0 a_{q,n,k}$**.**

(2) If $p = (s, S) \geq (t, T), (t, T')$**, then**

$$(t, T) \sim_{p,n+1} (t, T') \Leftrightarrow T \cap [S]_{n+1} = T' \cap [S]_{n+1}.$$

(3) If $a_{n,p,k} \geq_0 (t, T)$**, then** $a_{n,p,k} \sim_{p,n+1} (t, T)$**.**

So, $\{a_{p,n,k} \in \mathrm{P}_M \mid n \in \omega, p \in \mathrm{P}_M, 0 \leq k \leq f(p,n)\}$ **is a frame system for** P_M**.**

Laver forcing:

$p \geq_n q$ **iff** $p \geq q$ **and** $s_i^p = s_i^q$ **for all** $i = 0, \ldots, n+1$.

Let $f(p,n) = n$ **and** $K_n = \mathcal{P}(\{s_0^p, \ldots, s_n^p\})$.

For $p \in P_L$, $a_{p,n,k} = (p{\restriction} s_k^p) \setminus \{t \in p \mid \exists j > k[s_j^p \subseteq t]\}$.

If s_k^p **is** $\subseteq$**-maximal node among** K_n**, then** $a_{p,n,k} = p{\restriction} s_k^p$.

$stem_n(p) = \{t \in p \mid t \subseteq s_0^p\} \cup \{s_1^p, \ldots, s_{n+1}^p\} \cup \{t \in \omega^{<\omega} \mid s_{n+1}^p \prec t\}$.

Let $k = \max\{j \mid s_j^p \subsetneq s_{n+1}^p\}$. **Then, if** $k \neq j \leq n$**, then** $a_{stem_n(p),n+1,j} = a_{stem_n(p),n,j}$.

$a_{stem_n(p),n+1,k} = a_{stem_n(p),n,k} \setminus \{t \in stem_n(p) \mid s_{n+1}^p \subseteq t\}$.

Lemma.

(1) If $p \geq_n q$**, then** $a_{p,n,k} \geq_0 a_{q,n,k}$**.**

(2) If $p \geq r, r'$ **and** $r \sim_0 r'$**, then**

$$r \sim_{p,n+1} r' \Leftrightarrow r \cap K_{n+1} = r' \cap K_{n+1}.$$

(3) If $a_{n,p,k} \geq_0 r$**, then** $a_{n,p,k} \sim_{p,n+1} (t, T)$**.**

Lemma(Amalgamation Lemma). Let $(\mathrm{P}, \leq, \{\leq_n\}_{n\in\omega})$ be a fusion poset with a frame system which satisfies (C1), (C2) and (C3). If $n \in \omega, p, r \in \mathrm{P}$ and $r \leq_0 a_{p,n,k}$, then we have $a_{p|r,n,k} = r$.

Proof. Suppose that $n \in \omega, p, r \in \mathrm{P}$ and $r \leq_0 a_{p,n,k}$. Let $q = p|r$. Then we have

$$(*) \qquad \forall r' \in \mathrm{P}[q \geq r' \wedge r \sim_0 r' \wedge r \sim_{p,n+1} r' \Rightarrow r \geq r']$$

Since $\{a_{p,n,k}\}_{0\leq k\leq f(p,n)}$ is a partiotion of p and $r \leq a_{p,n,k}$, r is not compatible with $a_{p,n,j}$ for all $j \neq k$. By virtue

of (FS3), we have $a_{p,n,j} \geq a_{q,n,j}$. So r is not compatible with $a_{q,n,j}$ for all $j \neq k$.

Since $q = p|r \geq r$, we have $a_{q,n,k} \geq r$.

We have $a_{p,n,k} \geq_0 a_{q,n,k}$ by (FS3) , so that we have $r \sim_0 a_{q,n,k}$. Hence by virtue of (CS3), $a_{q,n,k} \geq_0 r$. Therefore $a_{q,n,k} \sim_{q,n+1} r$ by (FS7). Since $p \geq_n q$, we have $stem_n(p) = stem_n(q)$, so that $a_{q,n,k} \sim_{p,n+1} r$. So we have $r \geq a_{q,n,k}$ by $(*)$. Thus $a_{q,n,k} = a_{p|r,n,k} = r$.

Lemma. Let $(\mathrm{P}, \leq, \{\leq_n\}_{n\in\omega})$ be a fusion poset with a frame system which satisfies (C1), (C2) and (C3). Suppose that W is a partition of P and $p \in \mathrm{P}$. Then there exists $q \leq_0 p$ such that q is compatible with at most countably many $r \in W$.

Proof. It follows from the Amalgamation Lemma by usual arguments.

Theorem. If $(\mathrm{P}, \leq, \{\leq_n\}_{n\in\omega})$ is a fusion poset with a frame system which satisfies (C1), (C2) and (C3), then $(P, \leq, \{\leq_n\}_{n\in\omega})$ satisfies (A4).

Proof. By usual arguments.

Teorem. If $(\mathrm{P}, \leq, \{\leq_n\}_{n\in\omega})$ is a fusion poset with a frame system which satisfies (C1), (C2) and (C3), then $(P, \leq, \{\leq_n\}_{n\in\omega})$ satisfies (C4).

Proof. Let $\{a_{p,n,k}\}$ be a frame system for P, X be a pairwise incomparable subset of P such that $\forall r, r' \in X[r \sim_0 r']$ and $p \in \mathrm{P}$.

We shall show that there exists $q \leq_n p$ such that $r \not\leq q$ for all $r \in X$.

If there exists no $r \in X$ such that $r \leq p$, then we put $q = p$.

So we assume that there exists $r \in X$ such that $r \leq p$. Let $\ell = f(stem_n(p), n+1)$ and $\mathcal{P}(I^*_{p,n+1}) = \{t_1, ..., t_{2^\ell+1}\}$.

We construct a sequence $\{q_k\}_{0 \leq k \leq 2^{\ell+1}+1}$ inductively such that

$q_{k+1} \leq_n q_k$ for all k. Put $q_0 = p$.

Suppose that we already have q_k. In the following, we denote $\{j \mid r \uparrow a_{stem_n(p),n+1,j}\}$ by $C(r)$.

(1): If there exists $r \in X$ such that $r \leq q_k$ and $C(r) = t_k$. We pick such an element r and take $\tilde{r} < r$ *such that* $r \sim_{p,n+1} \tilde{r}$ by (FS6). Then put $q_{k+1} = q_k | \tilde{r}$.

(2): Othewise, put $q_{k+1} = q_k$.

Finally we put $q = q_{2^{\ell+1}+1}$.

By virtue of the definition, we have $q \leq_n p$. So we shall show that $q \not\geq r$ for all $r \in X$. Suppose that $q \geq r$ for some $r \in X$. Put $t = C(r)$. Then $t = t_k$ for some k. Thus we have $q_k \geq q \geq r$ and $C(r) = t_k$. So, by the definition of the sequece $\{p_k\}$, we have defined $q_{k+1} = q_k | \tilde{r}$ where $\tilde{r} < r^*, \tilde{r} \sim_{p,n+1} r^*$*and* $C(r^*) = t_k$ *for some* $r^* \in X$. Then $C(\tilde{r}) = C(r^*) = t_k = C(r)$. Since $q_k | \tilde{r} = q_{k+1} \geq q \geq r$, $\tilde{r} \geq r$ by (FS5). Hence we have $r^* > \tilde{r} \geq r$ and $r^*, r \in X$. This contradicts that X is a pairwise incomparable subset of P.

Teorem. Suppose that $(\mathrm{P}, \leq, \{\leq_n\}_{n\in\omega})$ is a fusion poset with a frame system which satisfies (C1), (C2) and (C3). Then, $(P, \leq, \{\leq_n\}_{n\in\omega})$ is not σ-short.

Finiteness Property

H. Mildenberger, The club principle and the distributivity number, Journal of Symbolic Logic, Vol. 76 No.1,2011, pp. 34-46

In this paper, Mildenberger defined the finiteness property for Axiom A posets. It is defined as follows.

Definition. An Axiom A poset $(\mathrm{P}, \leq, \{\leq_n\}_{n\in\omega})$ whose elements are subsets of $2^{<\omega}$ or of $\omega^{<\omega}$ has the finiteness property iff

1. $p \geq q$ implies $p \supseteq q$,
2. there is a function $h : \mathrm{P} \times \omega \longrightarrow \omega$ such that for every n, p, q,

$$p \geq_n q \text{ iff } p \geq q \text{ and } q \cap h(p,n)^{h(p,n)} = p \cap h(p,n)^{h(p,n)}.$$

In the case of $2^{<\omega}$, we can write $2^{h(p,n)}$ instead of $h(p,n)^{h(p,n)}$. We denote $2^{h(p,n)}$ or $h(p,n)^{h(p,n)}$ by H_n^p.

Without loss of generality, we may assume that elements of $\mathbb{P}$ are trees. We say that $\mathbb{P}$ has the **uniform finiteness property** if it has the finiteness property and **for every $n \in \omega, p, q \in \mathbb{P}$, $p \geq_n q$ implies $h(p,n) = h(q,n)$**.

For $p \in \mathbb{P}$, $s \in p$ is called **the stem of p** if

(i): for every $t \in p$, $s \subseteq t$ or $t \subseteq s$, and

(ii): p is a branching point, i.e., s has at least two successors in p.

We denote the stem of p as $st(p)$. If σ is a finite subtree of p, we denote it by $\sigma \Subset p$.

We say that $t \in \sigma$ is a σ-branching point of p if there exists $k \in \omega$ such that $t^\frown\langle k\rangle \in p$ and $t^\frown\langle k\rangle \notin \sigma$. We denote the set of σ-branching points of p by σ^b.

If $\mathrm{P} = P_L$, then $\sigma^b = \sigma \backslash \{t \in \sigma \mid t \subsetneq st(p)\}$

Let $p \geq r$ and $t \in \sigma^b$. Then we say that t is a r-σ-branching point of p if there exists $s \in r$ such that $t \subsetneq s$ and $\forall k \in \omega \left[t^\frown\langle k\rangle \subseteq s \Rightarrow t^\frown\langle k\rangle \notin \sigma\right]$. We donote the set of r-σ-branching points of p by $\sigma^{b,r}$. For $p \geq r, r'$ and $\sigma \Subset p$, we define $r \approx_\sigma r'$ if and only if $r \cap \sigma = r' \cap \sigma$ and $\sigma^{b,r} = \sigma^{b,r'}$.

We say that P has enough elements if P satisfies the following

1. $I = 2^{<\omega}$ or $\omega^{<\omega} \in \mathrm{P}$,
2. for every $r \in \mathrm{P}$, there exists $r' \in \mathrm{P}$ such that $r > r'$ and $st(r) = st(r')$,
3. for every $p \in \mathrm{P}$,
$$p^* = I \setminus \{t \in I \mid t \notin p, \exists s \in (H_n^p \setminus p)\ [s \subseteq t \text{ or } t \subseteq s]\} \in \mathrm{P},$$
4. for every $p \in \mathrm{P}$ and $s \in p$,
$$p{\restriction} s = \{t \in p \mid t \subseteq s \text{ or } s \subseteq t\} \in \mathrm{P},$$
5. for every $p \in \mathrm{P}$ and $r \leq p$,
$$p|r = r \cup \{t \in p \mid t \nsubseteq st(r) \text{ and } st(r) \nsubseteq t\} \in \mathrm{P}.$$

Let $\sigma_p^n = \{t \in \omega^{<\omega} \mid \exists s \in p \cap H_n^p\, [t \subseteq s]\}$**.**

Lemma. Let $(\mathrm{P}, \leq, \{\leq_n\}_{n\in\omega})$ **be an Axiom A poset with uniform finiteness property which has enough elements. Then for every** $n \in \omega, p \in \mathrm{P}$ **and** $p \geq r$**, there exists** $r' < r$ **such that** $r \approx_{\sigma_p^n} r'$**.**

Theorem. Let $(\mathrm{P}, \leq, \{\leq_n\}_{n\in\omega})$ be an Axiom A poset with uniform finiteness property which has enough elements. Then we have

1. P satisfies (C1), (C2) and (C3).
2. If $(\mathrm{P}, \leq, \{\leq_n\}_{n\in\omega})$ satisfies the following strong amalgamation property, then P is not σ-short.

$$(AP) : \forall n \in \omega \forall p \in \mathbf{P} \forall r \in \mathbf{P} \, [p \geq r \Rightarrow$$

$$\exists q \leq_n p \left[q \geq r \wedge \forall r' \in \mathbf{P} [q \geq r' \wedge r \approx_{\sigma_p^n} r' \Rightarrow r \geq r' \right]\right]$$

In the following, we assume that

(FS8): $\forall p \in \mathbf{P} \forall n \in \omega \forall k \in I^*_{p,n}$

$$\exists m \in \omega \exists j \in I_{1,m} \left[a_{stem_n(p),n,k} = a_{1,m,j}\right]$$

(FS9): $\forall p, r, r' \in \mathbf{P} \left[p \geq r, r' \Rightarrow [\forall n \in \omega [r \sim_{p,n} r'] \Rightarrow r = r']\right].$

(FS10): $\forall p \in \mathbf{P} \exists q \leq p\, [q$ **is uniform**$]\,,$

where q **is uniform if for every** $n \in \omega$**,**

$$\max\{m \mid \exists k, j \left[a_{stem_n(p),n,k} = a_{1,m,j}\right]$$
$$< \min\{m \mid \exists k, j \left[a_{stem_{n+1}(p),n+1,k} = a_{1,m,j}\right].$$

For every $p \in \mathrm{P}$, we define a subtree $\tilde{p}$ of $\omega^{<\omega}$ by

$$\tilde{p} = \{\tau \in \omega^{<\omega} \mid \forall n \in \mathrm{dom}(\tau)[0 \leq \tau(n) \leq f(1,n), p \upharpoonright a_{1,n,\tau(n)}]\}.$$

Let $\widetilde{\mathrm{P}} = \{\tilde{p} \mid p \in \mathrm{P}\}$. We define a partial order $\leq_T$ on $\widetilde{\mathrm{P}}$ such that $\tilde{p} \leq_T \tilde{q}$ if and only if $\tilde{p}$ is a subtree of $\tilde{q}$. We denote $I_{1,n}$ by I_n. For $n \in \omega$ and $j \in I_n$, we define $\tau^n_j \in \omega^{<\omega}$ by $\mathrm{dom}(\tau^n_j) = \{0, \ldots, n\}$ and $\tau^n_j(k) = \ell$ if and only if $a_{1,k,\ell} \geq a_{1,n,j}$. Since $\{a_{1,k,\ell}\}_{0 \leq \ell \leq f(1,k)}$ is a partition of 1 and $\{a_{1,n,j}\}_{0 \leq j \leq f(1,n)}$ is a refinement of $\{a_{1,k,\ell}\}_{0 \leq \ell \leq f(1,k)}$ for $k \leq n$, τ^n_j is well-defined. If $a_{stem_n(p),n,k} = a_{1,m,j}$, we denote τ^m_j by $\tau_{p,n,k}$.

Lemma. $\tilde{p}$ is a subtree of $\omega^{<\omega}$ for every $p \in \mathrm{P}$.

Lemma. For every $\tau \in \tilde{p}$, there are extensions τ_1, τ_2 of τ such that τ_1 and τ_2 are incompatible.

Lemma.

1. If $p \neq q$, then $\tilde{p} \neq \tilde{q}$.
2. If $p \leq q$, then $\tilde{p}$ is a subtree of $\tilde{q}$.
3. If $p \perp q$, then $\tilde{p} \perp \tilde{q}$.

Lemma. If $(\mathrm{P}, \leq)$ is a fusion poset with a frame system which satisfies (F8) and (F9), then $(\mathrm{P}, \leq)$ is isomorphic to $(\widetilde{\mathrm{P}}, \leq_T)$.

Let $\widetilde{\mathrm{P}}_u = \{\tilde{p} \mid p$ **is uniform**$\}$. **Then** $\widetilde{\mathrm{P}}_u$ **is a dense subset of** $\widetilde{\mathrm{P}}$ **by (F10).**

Theorem. If $\forall n \forall p \forall k [|\{j \mid a_{p,n,k} > a_{p,n+1,j}\}| = 2)]$, **then** $(\widetilde{\mathrm{P}}_u, \leq_T)$ **satisfies the finiteness property.**

It is open that $(\widetilde{\mathrm{P}}_u, \leq_T)$ **satisfies the finiteness property, in general.**

Remark. Since $\widetilde{P_L} \cong P_L$, $\widetilde{P_L}$ **satisfies the finiteness property as in [2].**

Open Problems

1. Hechler forcing which adds a strictly increasing function from ω to ω is not σ-short. How about general Hechler forcing?

2. Is a forcing product of a σ-closed poset and a CCC poset with the density $\geq \omega_1$ not σ-short?

3. Is Axiom A non-CCC poset not σ-short?

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